Recitation 6. April 13

Focus: computing determinants, Cramer's rule, diagonalization, eigenvalues and eigenvectors

There are three main ways of computing the determinant of an $n \times n$ matrix A:

- **row echelon form**: row reduce the matrix A, and then:

$$\boxed{\det A = \pm \text{ product of pivots}}$$

  where the sign is + if you did an even number of row exchanges, and − if you did an odd number of row exchanges.

- **the big formula**:

$$\det A = \sum_{\{\sigma(1),...,\sigma(n)\}}^{\text{permutations}} (-1)^{\text{sgn } \sigma} a_{1\sigma(1)} \dots a_{n\sigma(n)}$$

- **cofactor expansion**:

$$\begin{aligned} \text{along the } i\text{–th row:} \quad & \det A = a_{i1}C_{i1} + \dots + a_{in}C_{in} \\ \text{along the } i\text{–th column:} \quad & \det A = a_{1i}C_{1i} + \dots + a_{ni}C_{ni} \end{aligned}$$

  where $C_{ij} = (-1)^{i+j}$ times the determinant of the matrix obtained by removing row i and column j from A.

The formulas above also give rise to **cofactor formulas** for inverse matrices:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & \dots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \dots & C_{nn} \end{bmatrix}$$

The only formula for determinants that you may give without justification is the 2×2 case:

$$\boxed{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc}$$

Cramer's rule gives a quick formula for the solutions of a system $A\boldsymbol{v} = \boldsymbol{b}$ for an $n \times n$ matrix A:

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{where} \quad v_i = \frac{\det B_i}{\det A}$$

and B_i is obtained from A by replacing its i–th column with the vector $\boldsymbol{b}$.

To **diagonalize** a square matrix A means to write it as:

$$A = V \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} V^{-1}$$

Explicitly, the numbers $\lambda_1, ..., \lambda_n$ are called **eigenvalues** and the columns of V are called **eigenvectors**:

$$V = [\ \boldsymbol{v}_1 \mid \dots \mid \boldsymbol{v}_n\ ]$$

The way you compute these is the following. Eigenvalues are the roots of the characteristic polynomial:

$$\boxed{p(\lambda) = \det(A - \lambda I)}$$

Once you know the eigenvalues, the eigenvectors are computed as bases for nullspaces:

$$\boxed{A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i} \quad \Leftrightarrow \quad \boxed{\boldsymbol{v}_i \in N(A - \lambda_i I)}$$

1. Compute the determinant of:

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & -2 & 0 & 5 \\ -2 & 0 & -2 & 1 \\ 1 & 0 & -1 & 4 \end{bmatrix}$$

by doing a cofactor expansion along its second row.

Solution: Cofactor expansion tells us that the determinant of the matrix above equals:

$$(-1)^{2+1} \cdot 3 \det \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & 4 \end{bmatrix} + (-1)^{2+2} \cdot (-2) \det \begin{bmatrix} 1 & -1 & 0 \\ -2 & -2 & 1 \\ 1 & -1 & 4 \end{bmatrix} + (-1)^{2+4} \cdot 5 \det \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} =$$

$$-3(2((-2)(4)-(-1)))-2\big(((-2)(4)-(-1))+((-2)(4)-1)\big)+5((-1)(2)((-2)(-1)-(-2))) = 42+32-40 = 34.$$

2. Use the cofactor formula to invert the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Solution: First we compute the matrix of cofactors (since the cofactors are 2×2 determinants, we may compute them with the direct formula):

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 24 & 0 & 0 \\ -12 & 6 & 0 \\ -2 & -5 & 4 \end{bmatrix}$$

Moreover, the determinant of the original matrix is $1 \times 4 \times 6$ (product of pivots), so the inverse matrix is given by:

$$\frac{1}{24} \begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

(don't forget that the formula for the inverse matrix involves the transposed cofactor matrix).

3. Use Cramer's rule to solve the following system of equations:

$$\begin{cases} x + 3y - z = 0 \\ x + y + 4z = 0 \\ x + z = 1 \end{cases}$$

Solution: Let's first compute $\det \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix} = 11$ (you can get this in many ways, I personally recommend row reduction). Then Cramer's rule tells us that the solution for the system of equations is:

$$x = \frac{1}{11} \cdot \det \begin{bmatrix} 0 & 3 & -1 \\ 0 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix} = \frac{13}{11}$$

$$y = \frac{1}{11} \cdot \det \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix} = -\frac{5}{11}$$

$$z = \frac{1}{11} \cdot \det \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = -\frac{2}{11}$$

4. Find the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $\phi(v) = Av$. Can you find a basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ of $\mathbb{R}^3$ with respect to which ϕ is given by a diagonal matrix?

Solution: First we compute the characteristic polynomial:

$$det(A - \lambda I) = (2 - \lambda)(1 - \lambda)(-\lambda)$$

(we got this so easily because $A - \lambda I$ is an upper triangular matrix, so its determinant is the product of pivots). Thus, the eigenvalues of A are:

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$
$$\lambda_3 = 0$$

Eigenvectors for λ_1 are given by vectors in the nullspace of $A - 2I = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -2 \end{bmatrix}$, so an eigenvector is given by $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Eigenvectors for λ_2 are given by vectors in the nullspace of $A - 1I = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}$, so an eigenvector is given by $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Eigenvectors for λ_3 are given by vectors in the nullspace of $A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so an eigenvector is given by $\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Hence if we set:

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we have:

$$A = VDV^{-1}$$

By the change of basis formula, this means that with respect to the basis of eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, the linear transformation ϕ is given by $\text{diag}_{2,1,0}$.